

On Some Properties of Positive Definite Toeplitz Matrices and Their Possible Applications

Bishwa Nath Mukherjee
*Computer Science Unit
Indian Statistical Institute
Calcutta 700 035, India*

and

Sadhan Samar Maiti
*Department of Statistics
Kalyani University
Kalyani, Nadia, India*

Submitted by Miroslav Fiedler

ABSTRACT

Various properties of a real symmetric Toeplitz matrix Σ_m with elements $\sigma_{jk} = a_{|j-k|}$, $1 \leq j, k \leq m$, are reviewed here. Matrices of this kind often arise in applications in statistics, econometrics, psychometrics, structural engineering, multichannel filtering, reflection seismology, etc., and it is desirable to have techniques which exploit their special structure. Possible applications of the results related to their inverse, determinant, and eigenvalue problem are suggested.

1. INTRODUCTION

Most matrix forms that arise in the analysis of stationary stochastic processes, as in time series data, are of the so-called Toeplitz structure. The Toeplitz matrix, T , is important due to its distinctive property that the entries in the matrix depend only on the differences of the indices, and as a result, the elements on its principal diagonal as well as those lying within each of its subdiagonals are identical.

In 1910, O. Toeplitz studied various forms of the matrices $((t_{p-q}))$ in relation to Laurent power series and called them L -forms, without assuming that they are symmetric. A sufficiently well-structured theory on T -matrix

LINEAR ALGEBRA AND ITS APPLICATIONS 102:211-240 (1988)

211

© Elsevier Science Publishing Co., Inc., 1988
52 Vanderbilt Ave., New York, NY 10017

0024-3795/88/\$3.50

also existed in the memoirs of Frobenius [21, 22]. Although this matrix had its beginnings in pure mathematics, it has increasingly appeared in algebra, functional analysis, harmonic analysis, moment problems, probability theory, etc. The eigenvalue problems for the family $\{\mathbf{T}_m\}$ of Toeplitz matrices generated by a formal Laurent series of a rational function $R(Z)$ have engaged the attention of many mathematicians e.g., Day [14], Dickinson [16], Gorodeckii [29], Trench [68, 69]. The Toeplitz matrix has been also employed in a wide variety of applications, especially in the fields of numerical analysis, signal processing, system theory, etc. (Grenander and Szego [32], Basilevsky [6, pp. 219–223]). Citations of a large number of recent results have been made in the books of Iohvidov [42] and of Heining and Rost [39].

1.1. *The Model Generating the Symmetric Toeplitz Matrix*

For the purpose of understanding the properties of a symmetric Toeplitz matrix, consider a set of p measurements X_0, X_1, \dots, X_{p-1} , made at discrete equally spaced time points $t, t+1, \dots, t+(p-1)$ (possibly by sampling a continuous record). The correlation between X_t and any future value X_{t+k} is denoted as $\rho(X_t, X_{t+k}) = \rho(k)$. When the stochastic process is stationary, we have $\rho(X_t, X_{t+k}) = \rho(X_t, X_{t-k})$, or $\rho(k) = \rho(-k) = \rho_k$ for all $k = 1, 2, \dots, (p-1)$. Therefore the intercorrelation matrix ρ for the p variables will have a symmetric Toeplitz pattern when these variables follow a stationary (Gaussian) sequence.

It is to be noted here that the correlation function $\rho(k)$ does not depend on time t , but only on the time difference k . Correlations of the type $\rho(k)$ are also known as autocorrelation functions (Parzen [60]), since they reflect in a sense correlations on the same variable but between different time points, such as the height of an individual at different periods of his childhood. Such autocorrelation functions are generally observed in growth studies and in longitudinal studies of various psychological processes such as learning, forgetting, and development of abilities. In such processes, we frequently note a “growth” phenomenon which is invariant to a change in the time or space origin, and this is reflected in the autocorrelations of Toeplitz matrices.

1.2. *Main Purpose of the Paper*

In the field of statistical analysis, when the data are based on psychometric/biometric observations or time series in nature, statisticians very often face covariance matrices of the Toeplitz structure (Mukherjee and Maiti [56]). This is a particular case of the general Toeplitz matrix, viz. a positive definite (real) Toeplitz matrix with scalar elements. Such a square matrix has properties useful for deriving important inequalities and propositions. An attempt will be made in this paper to present some of these results with the

expectation that they might be useful in the theory of estimation and testing of statistical hypotheses related to Toeplitz structure of the population covariance matrix.

For the reasons mentioned above, we consider here the case of a symmetric positive definite Toeplitz matrix which can be regarded as the covariance operator for certain stationary processes occurring in discrete time. The symmetric Toeplitz operator endows it with a special structure which has been exploited in calculations such as solving linear equations and inversion of the matrix itself, in addition to fast matrix factorization and bounds on the spectral radius. Recent developments, together with some new results on these aspects, are briefly reviewed in this paper. Although the Toeplitz operator in continuous time processes is quite well known in astrophysics and other applied branches of mathematics, we will deal here with the discrete case.

Our main focus is on the statistical applications of the results. This is so because it is now well known that the growth curve can be estimated more efficiently and statistical tests of significance will be more powerful if the covariance or correlation structures arising in different repeated measurement designs (Winer [72]) are taken into account. The symmetric Toeplitz structure is a special case in this context. The results reported here are therefore expected to be useful in the estimation and testing of hypotheses in the area of stationary time series analysis of at least second order. Other possible applications of the results presented here are also discussed in the concluding section of this paper.

2. PRELIMINARY CONSIDERATIONS

We shall define here some specific types of matrices along with some of their important properties which will be helpful in the discussion of the symmetric Toeplitz matrices. We refer to Aitken [1] for this purpose.

2.1. *The Flip Matrix*

The flip matrix \mathbf{K}_m is an $m \times m$ matrix with 1's on the diagonal running southwest to northeast (which is at right angle to the principal diagonal, and is called the secondary diagonal) and 0's elsewhere. For example,

$$\mathbf{K}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrix \mathbf{K}_m reverses the order of the rows and columns of any matrix on pre- or postmultiplication. The flip operator \mathbf{K}_m has the following proper-

ties:

$$(1) \mathbf{K}_m = \mathbf{K}'_m = \mathbf{K}_m^{-1}.$$

(2) If $\mathbf{B}_{m \times n} = \mathbf{K}_m \mathbf{A}_{m \times n}$, then [i th row of \mathbf{B}] = [$(m - i + 1)$ th row of \mathbf{A}]. Similarly, if $\mathbf{C}_{m \times n} = \mathbf{A}_{m \times n} \mathbf{K}_n$, then [j th column of \mathbf{C}] = [$(m - j + 1)$ th column of \mathbf{A}]. And if $\mathbf{D}_{m \times n} = \mathbf{K}_m \mathbf{A}_{m \times n} \mathbf{K}_n$, then [(i, j) element of \mathbf{D}] = [$(m - i + 1, n - j + 1)$ element of \mathbf{A}].

2.2. Centrosymmetric Matrix

Any square matrix \mathbf{A} ($m \times m$) is called centrosymmetric if $\mathbf{K}_m \mathbf{A} \mathbf{K}_m = \mathbf{A}$. Thus if \mathbf{A} is centrosymmetric of order $m \times m$, we must have $a_{ij} = a_{m-i+1, m-j+1}$. A square matrix \mathbf{A} ($m \times m$) is centrosymmetric if and only if $\mathbf{K}_m \mathbf{A}$ is symmetric, or alternatively if and only if $\mathbf{A} \mathbf{K}_m$ is symmetric. Although many mathematicians (e.g., Huang and Cline [41]) treat persymmetry and centrosymmetry as identical, we will maintain a distinction following Aitken [1]. A square matrix \mathbf{A} is called persymmetric if $a_{ij} = a_{i+j-1}$. Thus, in a persymmetric matrix, the elements within each of the secondary diagonals running southwest to northeast must be identical (Aitken [1, p. 130]). Obviously, the inverse of a nonsingular centrosymmetric matrix is also centrosymmetric (i.e., symmetric about the center of its array of elements on both the diagonals), but not necessarily persymmetric.

2.3. Symmetric Toeplitz Matrix

A matrix Σ_m ($m \times m$) is called (real) symmetric Toeplitz matrix if its elements σ_{ij} obey the rule $\sigma_{ij} = a_{|i-j|}$ for all $i, j = 1, \dots, m$. The matrix Σ_m is a function of m parameters, i.e., $\Sigma_m = \Sigma_m(a_0, a_1, \dots, a_{m-1})$, and is actually centrosymmetric in form. A symmetric Toeplitz matrix can be written in explicit form as

$$\Sigma_m = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \\ a_1 & a_0 & a_1 & \cdots & a_{m-2} \\ a_2 & a_1 & a_0 & \cdots & a_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_0 \end{bmatrix}. \quad (2.1)$$

In its most general form, the $m \times m$ Toeplitz matrix has the following structure:

$$\mathbf{T}_m = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{-m+1} & a_{-m+2} & a_{-m+3} & \cdots & a_0 \end{bmatrix}, \quad (2.2)$$

where the elements of \mathbf{T}_m are such that $a_{ij} = a_{i-j}$. The nonsingular matrix \mathbf{T}_m as defined in (2.2) can be always factored as

$$\mathbf{T}_m = \mathbf{A}\mathbf{D}\mathbf{B}, \quad (2.3)$$

where \mathbf{D} is a suitable diagonal matrix, and \mathbf{A} and \mathbf{B} are two Vandermonde matrices, the rows of \mathbf{A} being $(1, 1, \dots, 1)$, $(\gamma_1^{-1}, \gamma_2^{-1}, \dots, \gamma_m^{-1})$, $(\gamma_1^{-2}, \gamma_2^{-2}, \dots, \gamma_m^{-2})$, and so on. The columns of \mathbf{B} are $(1, 1, \dots, 1)'$, $(\gamma_1, \gamma_2, \dots, \gamma_m)'$, $(\gamma_1^2, \gamma_2^2, \dots, \gamma_m^2)'$, and so on, where $\gamma_1, \gamma_2, \dots, \gamma_m$ are suitable nonzero numbers.

It can be seen from (2.1) that Σ_m allows the representation

$$\Sigma_m = a_0 \mathbf{I}_m + a_1(\mathbf{U} + \mathbf{U}') + a_2(\mathbf{U}^2 + \mathbf{U}'^2) + \dots + a_{m-1}(\mathbf{U}^{m-1} + \mathbf{U}'^{m-1}), \quad (2.4)$$

where $\mathbf{U} = ((u_{ij}))_{m \times m}$ (usually called the shift matrix) with $u_{ij} = 1$ for $i = j + 1$, $j = 1, \dots, m - 1$, and $= 0$ elsewhere. Equation (2.4) is a finite "power series expansion" of the matrix Σ_m (Whittle [70, p. 33]).

Another useful way of writing (2.1) is in terms of the $m \times m$ Toeplitz autocorrelation matrix ρ having the linear structure

$$\Sigma_m = a_0 \sum_{k=0}^{m-1} \rho_k \mathbf{H}_k = a_0 \rho_m, \quad (2.5)$$

where $a_0 \neq 0$, $\rho_k = a_k/a_0$, and each of the design matrices \mathbf{H}_k has components $h_{ij}^{(k)} = 1$ when $|i - j| = k$, and $= 0$ elsewhere.

The $m \times m$ symmetric Toeplitz correlation matrix ρ_m of (2.5) can also be written as the difference between the products of two sets of $m \times m$ triangular Toeplitz matrices of the form

$$\rho_m = \mathbf{G}\mathbf{G}' - (\mathbf{G} - \mathbf{I}_m)(\mathbf{G} - \mathbf{I}_m)', \quad (2.6)$$

where \mathbf{G} is a lower triangular matrix with unity in the principal diagonal and the other elements exactly the same as in the lower triangular part of the matrix ρ_m .

2.4. Hankel Matrix

Any matrix $\mathbf{F} = ((f_{ij}))_{m \times m}$ satisfying $f_{ij} = f_{i+j}$, $i, j = 0, 1, \dots, m - 1$, for some arbitrary numbers $f_0, f_1, \dots, f_{2m-2}$ is called a Hankel matrix (Gantmacher [24]).

A Hankel matrix is always persymmetric, but this is not true of a symmetric Toeplitz matrix. However, if \mathbf{F} ($m \times m$) is Hankel, $\mathbf{K}_m \mathbf{F}$ is nonsymmetric Toeplitz. Similarly, if $\mathbf{F} \mathbf{K}_m$ is Hankel, \mathbf{F} is nonsymmetric Toeplitz. The converse is also true. Thus, \mathbf{T} is a nonsymmetric Toeplitz iff $\mathbf{K}_m \mathbf{T}$ is Hankel.

2.5. Schur Complement

The matrix $\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B}$ or $\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C}$, appearing in the calculation of the determinant of the supermatrix \mathbf{E} by a partition method such as

$$\begin{aligned} \det(\mathbf{E}) &= \det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{D}) \det(\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C}) \\ &= \det(\mathbf{A}) \det(\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B}) \end{aligned} \quad (2.7)$$

when \mathbf{A} (or \mathbf{D}) is nonsingular, is called the Schur complement of \mathbf{A} (or \mathbf{D}) in \mathbf{E} . The reader is referred to Haynsworth [38] for the definition and to Ouellet [58] for a general discussion of Schur complements in statistics.

Huang and Cline [41] proved the theorem that the nonsingular \mathbf{E} of (2.7) is Toeplitz iff

- (1) \mathbf{E}^{-1} is centrosymmetric and
- (2) the Schur complement of \mathbf{A} (or \mathbf{D}) in \mathbf{E} is also centrosymmetric.

It can be also shown, following Haynsworth [38], that the inertia of a partitioned Toeplitz matrix as shown in (2.7) can be expressed as

$$\text{In}(\mathbf{E}) = \text{In}(\mathbf{A}) + \text{In}(\mathbf{S}) \quad (2.8)$$

where \mathbf{S} is the Schur complement of \mathbf{A} in \mathbf{E} , and the symbol In refers to the inertia.

3. SOME PROPOSITIONS ON TOEPLITZ MATRICES

Unless otherwise specified, we shall consider in this section Toeplitz matrices which are real and symmetric with the expression (2.1).

PROPOSITION 3.1. *Any principal submatrix of order $r \times r$ of a Toeplitz matrix of order $m \times m$, say Σ_m ($r < m$), is a Toeplitz matrix.*

Proof. This is so because omission of any rows and corresponding columns from a Toeplitz matrix preserves the Toeplitz property. ■

PROPOSITION 3.2. *A symmetric Toeplitz matrix Σ_m is always centrosymmetric, but the converse is not necessarily true.*

Proof. By definition of centrosymmetry, the (i, j) element of Σ_m must satisfy

$$\sigma_{ij} = \sigma_{m-i+1, m-j+1} \quad (3.1)$$

for all $i, j = 1, \dots, m$. The Toeplitz condition is a particular case of (3.1), since $\sigma_{ij} = a_{|i-j|} = a_{|m-i+1-m+j-1|} = \sigma_{m-i+1, m-j+1}$. The converse is not necessarily true. ■

We now note that premultiplication by the matrix K_m as defined in Section 2.1 merely reverses the order of the rows or columns of a symmetric matrix but keeps the symmetry invariant. Therefore, we have

$$K_m \Sigma_m = \Sigma_m K_m, \quad (3.2)$$

which leads to the result

$$K_m \Sigma_m K_m = \Sigma_m. \quad (3.3)$$

Using the lower shift matrix U , as defined in (2.4), Kailath, Kung, and Morf [50] defined the two displacement ranks of any $m \times m$ matrix S as

$$|S = S - USU' \quad (3.4)$$

and

$$|S = S - U'SU. \quad (3.5)$$

It has been proved by them that the difference of the ranks of $|S$ and $|S$ does not exceed two. Moreover, for a nonsingular S , $\text{rank}(|S) = \text{rank}(|S^{-1})$ and $\text{rank}(|S) = \text{rank}(|S^{-1})$.

In the case of Σ_m , all these ranks would be equal to two due to the fact that

$$\mathbf{U}\Sigma_m\mathbf{U}' = \text{diag}(0, \Sigma_{m-1}), \quad (3.6)$$

$$\mathbf{U}'\Sigma_m\mathbf{U} = \text{diag}(\Sigma_{m-1}, 0), \quad (3.7)$$

$$\mathbf{J}\Sigma_m = \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ a_1 & & & \\ \vdots & & \mathbf{0} & \\ a_{m-1} & & & \end{bmatrix}, \quad (3.8)$$

$$\mathbf{J}\Sigma_m = \begin{bmatrix} & & & a_{m-1} \\ & \mathbf{0} & & \vdots \\ & & & a_1 \\ a_{m-1} & \cdots & a_1 & a_0 \end{bmatrix}. \quad (3.9)$$

Defining a pair of k -displacement matrices of any $m \times m$ matrix \mathbf{S} by

$$\mathbf{J}_k\mathbf{S} = \mathbf{S} - \mathbf{U}^k\mathbf{S}\mathbf{U}^{k'} \quad (3.10)$$

and

$$\mathbf{J}_k\mathbf{S} = \mathbf{S} - \mathbf{U}^{k'}\mathbf{S}\mathbf{U}^k, \quad (3.11)$$

the corresponding pair of ranks may be called the k -displacement ranks of \mathbf{S} . Since $\mathbf{U}^k\Sigma_m\mathbf{U}^{k'} = \text{diag}(0, \dots, 0, \Sigma_{m-k})$ and $\mathbf{U}^{k'}\Sigma_m\mathbf{U}^k = \text{diag}(\Sigma_{m-k}, 0, \dots, 0)$, the k -displacement ranks of Σ_m are equal to $k+1$ for $k = 0, 1, 2, \dots, m-1$.

We may now characterize a Toeplitz matrix by a pair of centrosymmetric matrices in the following proposition.

PROPOSITION 3.3. *Let \mathbf{A}_{m-1} $[(m-1) \times (m-1)]$ be the leading principal submatrix obtained by deleting the last (or first) row and last (or first) column of a matrix \mathbf{A}_m ($m \times m$). Then \mathbf{A}_m is a Toeplitz matrix if and only if both \mathbf{A}_{m-1} and \mathbf{A}_m are centrosymmetric.*

Proof. The proof for sufficiency of the condition is obvious via Proposition 3.2. To prove the necessity of the condition, we proceed as follows:

As \mathbf{A}_{m-1} and \mathbf{A}_m are centrosymmetric,

$$a_{ij} = a_{m-i, m-j}, \quad i, j = 1, \dots, m-1, \quad (3.12)$$

and

$$a_{ij} = a_{m-i+1, m-j+1}, \quad i, j = 1, \dots, m. \quad (3.13)$$

Putting $j = i + k$, we obtain

$$a_{i, i+k} = a_{m-i+1, m-i+1-k} = a_{m-i, m-i-k}$$

and

$$a_{m-k, m} = a_{k+1, 1} \quad (3.14)$$

for all $i = 1, 2, \dots, m-1-k$, $k = 0, 1, 2, \dots, m-1$. From (3.14) we have, in addition to the symmetry of \mathbf{A}_m ,

$$a_{1, k+1} = a_{2, k+2} = \dots = a_{m-k, m}, \quad k = 0, 1, 2, \dots, m-1. \quad (3.15)$$

This ensures that \mathbf{A}_m is an $m \times m$ array of m distinct elements as in (2.1). ■

A characterization of a Toeplitz matrix similar to Proposition 3.3 has been obtained by Huang and Cline [41] in an alternative manner. They also obtained a simple criterion for recognizing an inverse of a matrix to be Toeplitz matrix.

Using the lower shift \mathbf{U} matrix of order $m \times m$ as defined in (2.4), we can give still another representation of the symmetric Toeplitz matrix as a sequel to Proposition 3.1. Let \mathbf{U}^* be the matrix \mathbf{U} after its last k columns have been deleted. It can be checked now that

$$\Sigma_{m-k} = \mathbf{U}^* \Sigma_m \mathbf{U}^*, \quad (3.16)$$

where Σ_{m-k} is the resultant symmetric Toeplitz matrix of dimension $m-k$ ($k = 0, 1, 2, \dots, m-1$). We shall give a general representation of the Toeplitz matrix (2.1) in the light of centrosymmetric matrix expressed in partitioned form (Aitken [1]). For even order,

$$\begin{aligned} \Sigma_{2m} &= \Sigma_{2m}(a_0, a_1, \dots, a_{m-1}, a_m, \dots, a_{2m-1}) \\ &= \begin{bmatrix} \Sigma_m & \Delta_m \\ \Delta'_m & \Sigma_m \end{bmatrix}, \end{aligned} \quad (3.17)$$

where $\Sigma_m = \Sigma_m(a_0, a_1, \dots, a_{m-1})$ and

$$\Delta_m = \begin{bmatrix} a_m & a_{m+1} & \cdots & a_{2m-1} \\ a_{m-1} & a_m & \cdots & a_{2m-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_m \end{bmatrix} = \mathbf{K}_m \Delta'_m \mathbf{K}_m. \quad (3.18)$$

For odd order,

$$\begin{aligned} \Sigma_{2m+1} &= \Sigma_{2m+1}(a_0, a_1, \dots, a_{m-1}; a_m; a_{m+1}, \dots, a_{2m}) \\ &= \begin{bmatrix} \Sigma_m & \mathbf{K}_m \underline{\beta}_m & \Delta_m^* \\ \underline{\beta}_m' \mathbf{K}_m & a_0 & \underline{\beta}_m' \\ \Delta_m^{*'} & \underline{\beta}_m & \Sigma_m \end{bmatrix} \end{aligned} \quad (3.19)$$

where $\underline{\beta}_m' = (a_1, a_2, \dots, a_{m-1} a_m)$ and

$$\Delta_m^* = \begin{bmatrix} a_{m+1} & a_{m+2} & \cdots & a_{2m} \\ a_m & a_{m+1} & \cdots & a_{2m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_{m+1} \end{bmatrix} = \mathbf{K}_m \Delta_m^{*'} \mathbf{K}_m. \quad (3.20)$$

4. DETERMINANT OF A SYMMETRIC TOEPLITZ MATRIX

Although Whittle [70, p. 39] gave a good approximation of the determinant of the matrix Σ_m as defined in (2.1), this cannot be used in the absence of raw data when we have to work with only the $m \times m$ symmetric Toeplitz correlation matrix ρ_m of (2.5). Daniels [13] has shown that $\det(\rho_m)$ can be expressed exactly in terms of partial correlations. If we use ρ_j to denote the partial correlation coefficient between X_t and X_{t+j} conditional on fixed $X_{t+1}, \dots, X_{t+j-1}$, which may be called the j th leading partial serial correlation coefficient, then it can be shown that

$$\begin{aligned} \det(\rho_m) &= (1 - \rho_1^2)^{m-1} (1 - \rho_{2\cdot}^2)^{m-2} \cdots (1 - \rho_{m-1\cdot}^2) \\ &= \prod_{j=1}^{m-1} (1 - \rho_j^2)^{m-j}. \end{aligned} \quad (4.1)$$

If the determinant of ρ_{m-1} and its inverse are already known, then it can be

checked that

$$\det(\rho_m) = \left(1 - \underline{\rho}' \rho_{m-1}^{-1} \underline{\rho}^2\right) \det(\rho_{m-1}), \quad (4.2)$$

where $\underline{\rho}^2 = (\rho_1, \rho_2, \dots, \rho_{m-1})'$. The result follows from the direct use of the Schur complement.

In order to calculate the $\det(\Sigma_m)$, we may apply the following three different approaches, each of which can be easily extended to block symmetric Toeplitz matrices.

4.1. Reduction by Splitting into Halves

Using a suitable matrix (Aitken [1]), we may split the determinant into a product of two determinants. Using the same notation as in (3.17), we can have the determinant for even order:

$$\det(\Sigma_{2m}) = \det(\Sigma_m + \Delta_m \mathbf{K}_m) \det(\Sigma_m - \Delta_m \mathbf{K}_m). \quad (4.3)$$

For odd order [vide (3.19)] we obtain

$$\det(\Sigma_{2m+1}) = \det \begin{bmatrix} \Sigma_m + \Delta_m \mathbf{K}_m & \sqrt{2} \mathbf{K}_m \underline{\beta}_m \\ \sqrt{2} \underline{\beta}_m' \mathbf{K}_m & a_0 \end{bmatrix} \det(\Sigma_m - \Delta_m \mathbf{K}_m). \quad (4.4)$$

Caflich [11] also proved essentially the same result but in a different manner. For example, his result for even order can be written as

$$\det(\Sigma_{2m}) = \det(\Sigma_m + \Delta_m' \mathbf{K}_m) \det(\Delta_m' \mathbf{K}_m - \Sigma_m). \quad (4.5)$$

4.2. Simple Reduction by Partitioning

For any order m ,

$$\begin{aligned} \det(\Sigma_m) &= \det \begin{bmatrix} \Sigma_{m-1} & \mathbf{K}_{m-1} \underline{\beta}_{m-1} \\ \underline{\beta}_{m-1}' \mathbf{K}_{m-1} & a_0 \end{bmatrix} \\ &= \det(\Sigma_{m-1}) \left(a_0 - \underline{\beta}_{m-1}' \Sigma_{m-1}^{-1} \underline{\beta}_{m-1} \right) \quad \text{provided } \Sigma_{m-1}^{-1} \text{ exists} \\ &= a_0 \det \left(\Sigma_{m-1} - \frac{\underline{\beta}_{m-1} \underline{\beta}_{m-1}'}{a_0} \right) \quad \text{provided } a_0 \neq 0. \end{aligned} \quad (4.6)$$

These are obtained by direct use of the Schur complement.

4.3. Reduction as a Function of $\det(\Sigma_{m-1})$ and $\det(\Sigma_{m-2})$

Using the Sylvester identity for border determinants (Iohvidov [42, p. 7]), we may express $\det(\Sigma_m)$ as

$$\det(\Sigma_m) \det(\Sigma_{m-2}) = [\det(\Sigma_{m-1})]^2 - [\det(\Phi_{m-1})]^2, \quad (4.7)$$

where

$$\Phi_{m-1} = \left[\begin{array}{c|cccccc} a_1 & a_0 & a_1 & a_2 & \cdots & a_{m-3} \\ a_2 & a_1 & a_0 & a_1 & \cdots & a_{m-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{m-2} & a_{m-3} & a_{m-4} & \cdots & a_0 & \\ \hline a_{m-1} & a_{m-2} & a_{m-3} & 1_{m-4} & \cdots & a_1 \end{array} \right],$$

so that

$$\det(\Phi_{m-1}) = (-1)^{m-2} \det(\Sigma_{m-2}) (a_{m-1} - \underline{\beta}'_{m-2} \Sigma_{m-2}^{-1} \mathbf{K}_{m-2} \underline{\beta}_{m-2}) \quad (4.8)$$

provided Σ_{m-2}^{-1} exists.

Computationally, the method of Section 4.1 seems to be the most efficient, as it requires considerably less time and labor than the methods discussed above. Once $\det(\Sigma_{m-1})$, Σ_{m-1}^{-1} , $\det(\Sigma_{m-2})$, etc. are known, the methods of Sections 4.2 and 4.3 may be used accordingly.

The procedure suggested in Section 4.3 has an additional utilization in the “nonsingular extension” of Σ_{m-1} to Σ_m . To make this possible, we require only one more element, e.g. a_{m-1} . Thus Σ_m will be a nonsingular extension of Σ_{m-1} provided

$$a_{m-1} \neq [\det(\Sigma_{m-2})]^{-1} [\pm \det(\Sigma_{m-1}) + \det(\Phi_{m-1}(0))], \quad (4.9)$$

where $\Phi_{m-1}(0)$ is the matrix in (4.4) with a_{m-1} replaced by zero. The expression (4.5) is simplified consequently to

$$a_{m-1} \neq \pm \frac{\det(\Sigma_{m-1})}{\det(\Sigma_{m-2})} + (-1)^{m-1} \underline{\beta}'_{m-2} \Sigma_{m-2}^{-1} \mathbf{K}_{m-2} \underline{\beta}_{m-2}. \quad (4.10)$$

The notion of nonsingular extension (Iohvidov [42]) leads to the following obvious proposition:

PROPOSITION 4.1. *It is possible to write out a sequence of n nonsingular Σ_i 's, $i = 1, 2, 3, \dots$, provided a sequence of elements a_{i-1} , $i = 1, 2, 3, \dots$, exists satisfying the inequalities*

$$a_0 \neq 0, \quad a_1 \neq \pm a_0, \\ a_i \neq \pm \frac{\det(\Sigma_i)}{\det(\Sigma_{i-1})} + (-1)^i \underline{\beta}'_{i-1} \Sigma_{i-1}^{-1} \mathbf{K}_{i-1} \underline{\beta}_{i-1}, \quad i = 2, 3, 4, \dots \quad (4.11)$$

4.4. Determinant as the Product of Determinants of Two Triangular Matrices

The symmetric Toeplitz matrix (2.1) can always be written in the form of the product of a lower and an upper triangular matrix using a Cholesky decomposition of the form

$$\Sigma_m = \mathbf{L}^* \mathbf{L}' = \mathbf{L} \mathbf{D} \mathbf{L}', \quad (4.12)$$

where \mathbf{D} is a diagonal matrix and the principal diagonal of \mathbf{L} consists of elements equal to unity. Then $\det(\Sigma_m)$ is the product of all the diagonal elements of \mathbf{D} (see, for example, Nehorai and Morf [57]). The Levinson-Durbin recursive algorithm provides a way of computing the successive rows of the Cholesky factor \mathbf{L} and the diagonal elements of \mathbf{D} .

We also note here that for any $\Sigma_m = a_0 \rho_m$, as defined in (2.5), we can always find, following Grunbaum [37], a unique tridiagonal matrix \mathbf{B} of the same order having a simple spectrum which commutes with Σ_m . The determinant and the eigenvalues as well as eigenvectors of the tridiagonal matrix \mathbf{B} are each equal to the corresponding quantities of Σ_m . Hence, once the problem of finding this matrix \mathbf{B} is solved, the evaluation of the determinant is done routinely and the solution of the eigenproblem corresponding to Σ_m is easily achieved.

5. INVERSION OF A NONSINGULAR TOEPLITZ MATRIX

The inversion of Toeplitz matrices has been approached in the literature (Trench [66, 67]; Justice [47]; Calderon, Spitzer, and Widom [12]; Widom

[71]; Gohberg and Feldman [25]; Heining and Rost [39]; Böttcher and Silbermann [9]) as a problem of finding an explicit form of the inverse and also for evaluating the computational merit of the algorithm used in the inversion. Levinson [52] showed in 1947 that a nonsingular $m \times m$ Toeplitz matrix with nonzero leading minors can be inverted with of the order of m^2 multiplication operations as compared to the order of m^3 multiplications generally required for a non-Toeplitz matrix. Gohberg and Semencul [27] have reviewed and established various methods of finding the inverse of a general Toeplitz matrix. Kailath, Vieira, and Morf [51] have also reviewed this field. Mentz [54] developed a procedure to find the components of the inverse matrix in closed form when the $m \times m$ matrix has only $2p + 1$ nonvanishing (central) diagonals ($1 \leq p < m$). The procedure consists in posing difference equations for the components of the inverse and solving them explicitly.

Trench [66] gave a recursive formula for a nonsingular scalar entried symmetric correlation matrix of Toeplitz form (2.5) which has since been rederived, elaborated, and extended by several workers, particularly Akaike [3] and Zohar [75, 76]. The formula for computing the inverse of the correlation matrix ρ_m is

$$[\rho_m^{-1}]_{i+1, j+1} = [\rho_m^{-1}]_{i, j} + \frac{1}{K_0} \{ b_{i, m} c_{j, m} - c_{m+1-i, m} b_{m+1-j, m} \},$$

where the coefficients K_0 , $\{b_{i, m}\}$, $\{c_{i, m}\}$ are found from certain simple equations, such as the Yule-Walker equations (Pagano [59]), which can be efficiently solved by the Levinson-Durbin algorithm (Levinson [52]; Durbin [19]). Wise [73] has given a method of finding ρ_m^{-1} of (2.5) based on the spectral density function. Exploiting the symmetry, Siddiqui [65] proposed an alternative procedure of computing the inverse of a variance-covariance matrix for an autoregressive scheme of a given order.

We shall develop here a few additional procedures which might be helpful due to their being less complicated.

5.1. Inverse as the Difference of Two Matrices

Analogous to (2.6), the inverse matrix can be written as

$$\Sigma_m^{-1} = \alpha_0^{-1}(\mathbf{BB}' - \mathbf{CC}'), \quad (5.1)$$

where

$$\mathbf{B}_{m \times m} = \begin{bmatrix} \alpha_0 & & & & 0 \\ \alpha_1 & \alpha_0 & & & \\ \alpha_2 & \alpha_1 & \alpha_0 & & \\ \vdots & & & & \\ \alpha_{m-1} & \alpha_{m-2} & \alpha_{m-3} & \cdots & \alpha_0 \end{bmatrix},$$

$$\mathbf{C}_{m \times m} = \begin{bmatrix} 0 & & & & 0 \\ \alpha_{m-1} & 0 & & & \\ \alpha_{m-2} & \alpha_{m-1} & 0 & & \\ \vdots & & & & \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{m-1} 0 \end{bmatrix},$$

$$\alpha_0 = \frac{\det(\Sigma_{m-1})}{\det(\Sigma_m)},$$

and

$$\underline{\delta}_{m-1} = (\alpha_1 \alpha_2 \cdots \alpha_{m-1})' = -\alpha_0 \Sigma_{m-1}^{-1} \underline{\beta}_{m-1}$$

provided $\det(\Sigma_m) \neq 0$ and $\det(\Sigma_{m-1}) \neq 0$. Essentially the same result has been reported by Kailath, Buckstein, and Morgan [48]. Kailath, Vieira, and Morf [51] credited this formula to Gohberg and Semencul [27].

PROPOSITION 5.1. *The inverse of a nonsingular Toeplitz matrix is a centrosymmetric matrix.*

Proof. Owing to centrosymmetric property of Σ_m , the cofactor of the (i, j) element of Σ_m equals the cofactor of its $(m-i+1, m-j+1)$ element. This means $\Sigma_m^{-1} = \mathbf{K}_m \Sigma_m^{-1} \mathbf{K}_m$, as the (i, j) element of Σ_m^{-1} equals the cofactor of the (j, i) element of Σ_m divided by $\det(\Sigma_m)$. ■

5.2. Inversion by Splitting into Halves

For even order Σ_{2m} [vide (3.17)], consider an orthogonal matrix

$$\mathbf{P}_{2m} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_m & \mathbf{K}_m \\ \mathbf{I}_m & -\mathbf{K}_m \end{bmatrix} \quad (5.2)$$

to get

$$\mathbf{P}_{2m} \Sigma_{2m} \mathbf{P}'_{2m} = \begin{bmatrix} \Sigma_m + \Delta_m \mathbf{K}_m & \mathbf{0} \\ \mathbf{0} & \Sigma_m - \Delta_m \mathbf{K}_m \end{bmatrix}. \quad (5.3)$$

Taking the inverse, we obtain

$$\mathbf{P}_{2m} \Sigma_{2m}^{-1} \mathbf{P}'_{2m} = \begin{bmatrix} (\Sigma_m + \Delta_m \mathbf{K}_m)^{-1} & \mathbf{0} \\ \mathbf{0} & (\Sigma_m - \Delta_m \mathbf{K}_m)^{-1} \end{bmatrix}. \quad (5.4)$$

Writing $\Sigma_{2m}^{-1} = \mathbf{B}$, we may express \mathbf{B} in partitioned form utilizing Proposition 5.1 to give its general representation as

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{K}_m \mathbf{B}_{12} \mathbf{K}_m & \mathbf{K}_m \mathbf{B}_{11} \mathbf{K}_m \end{bmatrix}. \quad (5.5)$$

Then using (5.2), we find

$$\mathbf{P}_{2m} \mathbf{B} \mathbf{P}'_{2m} = \begin{bmatrix} \mathbf{B}_{11} + \mathbf{B}_{12} \mathbf{K}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{11} - \mathbf{B}_{12} \mathbf{K}_m \end{bmatrix}. \quad (5.6)$$

Comparing (5.6) and (5.4), we solve for \mathbf{B}_{11} and \mathbf{B}_{12} as

$$\mathbf{B}_{11} = \frac{1}{2} \left[(\Sigma_m + \Delta_m \mathbf{K}_m)^{-1} + (\Sigma_m - \Delta_m \mathbf{K}_m)^{-1} \right], \quad (5.7)$$

$$\mathbf{B}_{12} = \frac{1}{2} \left[(\Sigma_m + \Delta_m \mathbf{K}_m)^{-1} - (\Sigma_m - \Delta_m \mathbf{K}_m)^{-1} \right] \mathbf{K}_m. \quad (5.8)$$

Thus, for even order Σ_{2m} , Σ_{2m}^{-1} may be expressed as (5.5) with the aid of (5.7) and (5.8).

Similarly, for odd order Σ_{2m+1} [vide (3.19)], we use the orthogonal matrix

$$\mathbf{P}_{2m+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} & \mathbf{K}_m \\ \mathbf{0}' & \sqrt{2} & \mathbf{0}' \\ \mathbf{I}_m & \mathbf{0} & -\mathbf{K}_m \end{bmatrix} \quad (5.9)$$

to get Σ_{2m+1}^{-1} expressed as

$$\mathbf{B} = \left[\begin{array}{cc|c} \mathbf{B}_{11} & \varepsilon & \mathbf{B}_{12} \\ \varepsilon' & b_{m+1, m+1} & \varepsilon' \mathbf{K}_m \\ \hline \mathbf{K}_m \mathbf{B}_{12} \mathbf{K}_m & \mathbf{K}_m \varepsilon & \mathbf{K}_m \mathbf{B}_{11} \mathbf{K}_m \end{array} \right], \quad (5.10)$$

in which \mathbf{B}_{11} , \mathbf{B}_{12} , ε , and $b_{m+1, m+1}$ are to be found from

$$\left[\begin{array}{cc} \mathbf{B}_{11} + \mathbf{B}_{12} \mathbf{K}_m & \sqrt{2} \varepsilon \\ \sqrt{2} \varepsilon' & b_{m+1, m+1} \end{array} \right] = \left[\begin{array}{cc} \Sigma_m + \Delta_m^* \mathbf{K}_m & \sqrt{2} \mathbf{K}_m \beta_m \\ \sqrt{2} \beta_m' \mathbf{K}_m & a_0 \end{array} \right]^{-1} \quad (5.11)$$

and

$$\mathbf{B}_{11} - \mathbf{B}_{12} \mathbf{K}_m = (\Sigma_m - \Delta_m^* \mathbf{K}_m)^{-1}. \quad (5.12)$$

6. BOUND ON THE DETERMINANT OF A POSITIVE DEFINITE TOEPLITZ MATRIX

We shall give here, in the form of lemmas, various results related to bounds on $\det(\Sigma_m)$ when Σ_m is a positive definite (p.d.) symmetric Toeplitz matrix of order $m \times m$.

LEMMA 6.1. *The sequence $\{\det(\Sigma_i)/\det(\Sigma_{i-1}), i = 2, 3, \dots\}$ is a decreasing sequence of positive numbers less than a_0 (> 0).*

Proof. From the Sylvester identity (4.7), we get

$$\det(\Sigma_m) \det(\Sigma_{m-2}) \leq [\det(\Sigma_{m-1})]^2. \quad (6.1)$$

As Σ_m is p.d., Σ_{m-1} , Σ_{m-2} are also p.d., so that

$$0 < \frac{\det(\Sigma_m)}{\det(\Sigma_{m-1})} \leq \frac{\det(\Sigma_{m-1})}{\det(\Sigma_{m-2})}. \quad \blacksquare$$

LEMMA 6.2.

$$\det(\Sigma_m) < (1 - a_1^2/a_0^2)^{m-1} a_0^m. \quad (6.2)$$

Proof. From Lemma 6.1,

$$\begin{aligned} \frac{\det(\Sigma_m)}{\det(\Sigma_1)} &= \prod_{i=2}^m \frac{\det(\Sigma_i)}{\det(\Sigma_{i-1})} \\ &\leq \left(\frac{\det(\Sigma_2)}{\det(\Sigma_1)} \right)^{m-1} \\ &= \left(\frac{a_0^2 - a_1^2}{a_0} \right)^{m-1}. \quad \blacksquare \end{aligned}$$

LEMMA 6.3.

$$\det(\Sigma_{2m}) \leq \prod_{j=0}^{m-1} (a_0^2 - a_{2m-2j-1}^2) \quad (6.3)$$

and

$$\det(\Sigma_{2m+1}) \leq a_0 \prod_{j=0}^{m-1} (a_0^2 - a_{2m-2j}^2). \quad (6.4)$$

Proof. To prove these results, it is sufficient to note a well-known inequality from Bellman [7, p. 129]:

If \mathbf{W} ($m \times m$) is a p.d. matrix, $\det(\mathbf{W}) \leq \prod_{i=1}^n w_{ii}$. The equality holds when \mathbf{W} is a diagonal matrix.

This may be applied to p.d. matrices $\Sigma_m \pm \Delta_m \mathbf{K}_m$; vide (4.3) and (4.4). ■

LEMMA 6.4.

$$\sup_{p.d. \Sigma_m} \det(\Sigma_m) = a_0^m \left(1 - \frac{a_{m-1}^2}{a_0^2} \right). \quad (6.5)$$

Proof. Applying the condition of equality (from the abovementioned inequality in the proof of Lemma 6.3) to $\Sigma_m \pm \Delta_m \mathbf{K}_m$, we get the off-diagonal elements zero, so that

$$a_1 \pm a_{2m-2} = a_2 \pm a_{2m-3} = \cdots = a_{m-1} \pm a_m = 0.$$

Therefore, $a_1 = a_2 = \cdots = a_{2m-2} = 0$. Consequently, the uppermost bound on $\det(\Sigma_{2m})$ in (6.3) is

$$(a_0^2 - a_{2m-1}^2) a_0^{2(m-1)}.$$

Similarly, for odd order Σ_{2m+1} , the upper most bound on the determinant is $(a_0^2 - a_{2m}^2) a_0^{2m-1}$. ■

LEMMA 6.5.

$$\det(\Sigma_m) < \det(\bar{\Sigma}_m) \quad (6.6)$$

where $\bar{\Sigma}_m$ is an $m \times m$ matrix with a_0 as diagonal elements and $[(m-1)a_1 + (m-2)a_2 + \cdots + a_{m-1}]/[m(m-1)/2]$ ($= b$, say) as off-diagonal elements, so that

$$\det(\bar{\Sigma}_m) = (a_0 - b)^m [a_0 + (m-1)b]. \quad (6.7)$$

Proof. The reader is referred to Aitkin et al. [2] for a proof. ■

LEMMA 6.6. If ρ_m is the $m \times m$ symmetric correlation matrix of Toeplitz form as defined in (2.5) and $\det(\rho_m^*)$ is the determinant of the cofactor of the element in the first column and m th row of ρ_m , then the sequence $\{[\det(\rho_i^*)]^2/[\det(\rho_{i-1})]^2, i = 2, 3, \dots\}$ is an increasing sequence, and the index

$$\pi_{i-1} = \frac{\det(\rho_i^*)}{\det(\rho_{i-1})} \quad (6.8)$$

lies between -1 and $+1$.

Proof. This is a clear application of the Sylvester identity to ρ_m as shown in (4.7) and Lemma 6.1. ■

7. BOUNDS ON EIGENVALUES

The problem of obtaining an explicit solution for the eigenvalues of general Toeplitz matrices has been considered by various researchers. Whittle [70] made an attempt to evaluate those of Σ_m , but he obtained only an approximate result. Grunbaum [36, 37] has also considered the eigenvalue problem for general Toeplitz matrices. There is an extensive literature on Toeplitz matrices generated by rational functions, devoted mainly to studying the asymptotic distributions of the eigenvalues of the matrix as $m \rightarrow \infty$ (e.g. Dickinson [16]).

We shall present here some bounds on the eigenvalues ($\lambda_i > 0$, $i = 1, \dots, m$) of a positive definite matrix Σ_m . These results can be derived without much difficulty. For proofs, we shall only refer to original sources. Some of the general results are to be found in the article by Brauer [8], who has investigated the regions in which the eigenvalues of an arbitrary square matrix must lie. We also refer to the Perron-Frobenius theorem (Gantmacher [24, p. 65]), which guaranteed that any nonnegative matrix has an eigenvalue of maximum magnitude that is real and positive. In general, we define the maximum magnitude of eigenvalue of a square matrix as its spectral radius. We also keep in view Gerschgorin's theorem (see Pullman [62]) that for A $m \times m = ((a_{jk}))$ every eigenvalue lies in the union of the m closed intervals

$$\left[a_{jj} - \sum_{j \neq k} |a_{jk}|, a_{jj} + \sum_{j \neq k} |a_{jk}| \right], \quad j = 1, \dots, m. \quad (7.1)$$

Hoffman [40] also proved the theorem that every eigenvalue of A lies in the union of the m closed intervals

$$\left[\sum_k a_{jk} - m \left(\max_{j \neq k} a_{jk} \right)_+, \sum_k a_{jk} - m \left(\min_{j \neq k} a_{jk} \right)_- \right], \quad j = 1, \dots, m, \quad (7.2)$$

where $a_+ = |a|$ if $a > 0$ and $= 0$ otherwise; similarly, $a_- = -|a|$ if $a < 0$ and $= 0$ otherwise. These results can be simplified for the Toeplitz matrix Σ_m . In addition, we present below the following results in the form of propositions.

PROPOSITION 7.1.

$$0 < \lambda_{\min} \leq a_0 + \frac{2}{m} \sum_{i=1}^{m-1} (m-i)a_i \leq \lambda_{\max}. \quad (7.3)$$

Proof. The reader is referred to Pullman [62, p. 232] or Graybill [31, p. 312] for the proof. ■

PROPOSITION 7.2.

$$0 \leq \lambda_{\min} \leq a_0 \leq \lambda_{\max} \leq ma_0. \quad (7.4)$$

Proof. See Bellman [7, p. 41] for the proof. ■

PROPOSITION 7.3.

$$mb_m \leq \lambda_{\max} \leq ma_0, \quad (7.5)$$

where $b_m = \min_{0 \leq i \leq m-1} a_i$.

Proof. Apply (7.3) and (7.4) to obtain (7.5). ■

PROPOSITION 7.4.

$$0 \leq \lambda_{\min} \leq a_0 - \max |a_i|, \quad (7.6)$$

where the maximum is taken over $i = 1, 3, 5, \dots, 2m-1$ for even order Σ_{2m} and $i = 2, 4, 6, \dots, 2m$ for odd order Σ_{2m+1} .

Proof. As eigenvalues are similarity invariant, consider the eigenvalues of $\mathbf{P}_{2m} \Sigma_{2m} \mathbf{P}'_{2m}$ [vide (5.3)] and apply the theorem that the minimum eigenvalue is less than the minimum diagonal element. ■

PROPOSITION 7.5.

(a) We have

$$\prod_{i=1}^m \lambda_i \leq a_0^m, \quad \bar{\lambda} = \frac{1}{m} \sum_{i=1}^m \lambda_i = a_0. \quad (7.7)$$

(b) If $R_\lambda = (\text{range of eigenvalues}) = \lambda_{\max} - \lambda_{\min}$,

$$0 \leq \frac{R_\lambda}{\bar{\lambda}} \leq m. \quad (7.8)$$

Proof. The statements in (7.7) are obvious. The inequality (7.4) may be used to show (7.8). ■

PROPOSITION 7.6.

$$a_0 \in (\lambda_{\min}, \lambda_{\max}) \text{ and } a_j \in (-jR_\lambda, jR_\lambda) \text{ } j = 1, 2, \dots, m-1. \quad (7.9)$$

Proof. See Graybill [31, p. 314] for a proof. ■

PROPOSITION 7.7. If $a_0 > |a_1| > |a_2| > \dots > |a_{m-1}|$,

$$0 \leq \lambda_i \leq a_0 + (m-1)|a_i| \quad (7.10)$$

for all $i = 1, 2, \dots, m$.

Proof. Using Gerschgorin's theorem [vide (7.1)] and (7.4), we can prove (7.10). ■

Following are some additional inequalities related to the monotone behavior of the eigenvalues of Σ_m of different orders. These are mainly based on the Sturmian separation theorem (Bellman [7, p. 117]).

7.1 Sturmian Separation Theorem

Let $\lambda_k(A_r)$, $k = 1, \dots, r$, $r = 1, 2, \dots$, be the eigenvalues of symmetric matrices A_r ($r \times r$), and $\lambda_1(A_r) \geq \lambda_2(A_r) \geq \dots \geq \lambda_r(A_r)$. Then

$$\lambda_{k+1}(A_{r+1}) \leq \lambda_k(A_r) \leq \lambda_k(A_{r+1}) \quad \text{for any } k \leq r. \quad (7.11)$$

The inequalities in (7.11) naturally hold for Σ_m also. Furthermore, we get other inequalities as an immediate consequence:

$$a_0 = \lambda_{\max}(\Sigma_1) \leq \lambda_{\max}(\Sigma_2) \leq \dots \leq \lambda_{\max}(\Sigma_m) \leq ma_0, \quad (7.12)$$

$$a_0 = \lambda_{\min}(\Sigma_1) \geq \lambda_{\min}(\Sigma_2) \geq \dots \geq \lambda_{\min}(\Sigma_m) \geq 0, \quad (7.13)$$

$$0 = R_\lambda(\Sigma_1) \leq R_\lambda(\Sigma_2) \leq \dots \leq R_\lambda(\Sigma_m) \leq ma_0, \quad (7.14)$$

where $R_\lambda(\Sigma_i) = \lambda_{\max}(\Sigma_i) - \lambda_{\min}(\Sigma_i)$. Using (7.3) and (7.13), we establish

further that

$$0 \leq \lambda \max(\Sigma_m) - \lambda \max(\Sigma_{m-1}) \leq (m-1)R_a(\Sigma_{m-1}), \quad (7.15)$$

where $R_a(\Sigma_{m-1}) = \max(a_0, a_1, \dots, a_{m-2}) - \min(a_0, \dots, a_{m-2})$.

8. POSSIBLE APPLICATIONS

It is well known that statistical tests of significance can be seriously affected by serial correlation, i.e., the correlation which is usually present when data are collected in time. When testing for change in data observed at equally spaced time points, it is usually necessary to assume a time series model to represent and explain the serial correlations. One of the finite parameter models for stationary time series process which can give rise to the symmetric Toeplitz covariance matrix of the form (2.1) is the one in which the covariance between any two measurements X_t and X_{t+j} depends only on the time interval j . Hence there is justification for studying the properties of a symmetric Toeplitz matrix. It has been shown by Wise [73] that the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ of the autocovariance matrix Σ_m as defined in (2.1) will trace out the so-called spectral curve (density) of the time process that has generated the observed time series X_t . Wise [73] also showed that each λ_i of Σ_m can be expanded in a finite Fourier series.

The mathematical properties of Σ_m , as discussed in this paper, have interesting applications not only in the area of spectral analysis of time series, but also for various statistical applications. The symmetric Toeplitz matrix arises frequently in statistical work connected with stationary stochastic process, nonparametric theory, and repeated measurements of autoregressive processes. Granander and Szego [32, p. 231] have given a simple example illustrating how Toeplitz matrices appear in the theory of homogeneous crystal structures, specifically their motion. Knowing the elements of the symmetric Toeplitz matrix, it is possible to find the central mass sequence $(\gamma_1, \gamma_2, \dots, \gamma_m)$ for Σ_m defined as $\gamma_k = \sup\{\gamma: (\Sigma_k - \gamma J_k) \text{ is p.d.}\}$, $k = 1, \dots, m$, in which J_k is the $k \times k$ matrix with all elements unity. Caflish [11] has shown how this knowledge helps in solving direct and inverse problems of transmission lines which consist of piecewise constant components. Using the Schur algorithm, Caflish [11] has shown an alternative method of computing the reflection coefficient from the Toeplitz matrix. This method has also interesting network-theoretic and stochastic interpretations. Using the elements of the Cholesky decomposition factor L as shown in (4.12), Dickinson

[17] has suggested another method of computing the reflection-coefficient sequence. For a discussion, see Dewilde, Vieira, and Kailath [15].

Specific forms of the Toeplitz matrix lead to particularly simple expressions for eigenvalues and eigenvectors, many of which are conventionally applied in the analysis of time series data as well as in structural engineering.

A symmetric Toeplitz matrix of the form (2.1) cannot be reduced to a diagonal (canonical) form by pre- and postmultiplication of a transformation matrix the elements of which are independent of Σ_m . However, some of the results reported here are expected to help in checking whether an empirical time series can be fitted by an autoregressive process of a given order and also in the estimation and testing of the autocovariance structure (2.1), as evident from the recent work of the authors [56].

Most of the previous efforts in the area of estimation of time series analysis were devoted to the estimation of the parameters of autoregressive models. However, following Anderson [4] and Parzen [61], we can take as our parameters the autocovariance matrices of the observable random variables. The results are also useful in the analysis of covariance structures (Mukherjee [55], Jöreskog [46]) and structural spectral density matrices (Lillestol [53]) of the Toeplitz form, which in econometrics are frequently referred to as Laurent matrices (Whittle [70]). A special case that arises in exponential smoothing applications as used in business forecasting, for example, is the one in which $\Sigma_m = \rho_m$; $\sigma_0 = 1$, $\sigma_j = \rho^j$, $j = 1, 2, \dots, m-1$; and $0 < |\rho| < 1$. Grenander and Szego [32] have given an extended discussion of the properties and application of such matrices.

Since Toeplitz matrices of order m can be easily obtained from Hankel matrices by post- or premultiplication by a constant K_m matrix as shown in Section 2.4, all results about quasidirect decomposition of Hankel matrices correspond to results about quasidirect decomposition of Toeplitz matrices and conversely (Fiedler [20]). This result, together with the fact that a symmetric positive definite Toeplitz matrix of the form (2.1) can be uniquely factored with $O(m^2)$ element operations (Bareiss [5]; Rissanen [63]), may have important implications for any new computational algorithm relating to Toeplitz matrices. Zellini [74] has furthermore shown that the matrix product $A\bar{b}$ over a real field when A is an $m \times m$ symmetric Toeplitz matrix of the form (2.5) requires at least $2(m-1)$ multiplications, since the matrix A has tensor rank $2(m-1)$ in the real field. This has important implications for the computation of a finite set of bilinear forms formulated as a matrix product.

The Toeplitz covariance structure has been generalized to the block form also (Akaike [3]), and recently Kailath and Kolitracht [49] have obtained conditions for a nonsingular matrix to have a block Toeplitz inverse. Greville and Trench [35] found conditions for a nonsingular band matrix to have an inverse with the Toeplitz structure. The results of Kailath and Kolitracht [49]

have however generalized those of Huang and Cline [41] for Toeplitz matrices with scalar entries, and also relate to some results of Gohberg and Feldman [25] and Gohberg and Heining [26] as well as Greville [34]. Some of the known results for Toeplitz matrices have been generalized to conjugate-Toeplitz matrices (Gover and Barnett [30]).

As is well known, the evaluation of the determinant, the eigenvalues, and the inverse of the population covariance matrix finds wide application in statistics, both in multivariate analysis and in deriving theoretical properties of estimation and constructing new statistical tests of significance (Basilevsky [6]). For example, the asymptotic properties of Σ_m^{-1} as m tends to infinity have been considered and exploited by Grenander and Rosenblatt [33]. Such problems of linear algebra have also found wide application in structure mechanics and electrical engineering. The results concerning inversion of a symmetric Toeplitz matrix are useful, for example, not only in solving the Yule-Walker equation for estimating the parameters of the autocorrelation model (see Durbin [18], p. 312), but also for obtaining the solution of the central mass equations in electrical engineering (see Cafilisch [11]), statistical signal processing, system theory, and discrete inverse scattering (Gohberg, Kailath, and Kolitracht [28]). In the problem of discretization of the Copinath-Sondhi integral equation, the inversion of the symmetric Toeplitz matrix is required for solving the systems given by

$$\Sigma_\nu \varphi_\nu = \underline{1}, \quad 1 \leq \nu \leq m, \quad (8.1)$$

where Σ_ν is the $\nu \times \nu$ principal submatrix of Σ_m defined in (2.1), φ_ν is ν -component column vector of unknowns, and $\underline{1}_\nu$ is the ν -component column vector of unity. This is one of the important aspects of the one-dimensional inverse problem of reflection seismology (Bube and Burridge [10]). The results concerning the inverse as the difference of products of Toeplitz matrices as shown in (5.1) has furthermore some implications for interpreting and computing bilinear forms such as $\underline{y}' \mathbf{R}^{-1} \underline{x}$ where \underline{y} and \underline{x} are given vectors. Kailath, Vieira, and Morf [51] have shown that when \mathbf{R} is a Toeplitz matrix, then

$$\underline{y}' \mathbf{R}^{-1} \underline{x} = (\mathbf{L}'_1 \underline{y})' (\mathbf{U}_1 \underline{x}) - (\mathbf{L}'_2 \underline{y})' (\mathbf{U}_2 \underline{x}) \quad (8.2)$$

where $\{\mathbf{L}_j\}$, $\{\mathbf{U}_j\}$ are lower and upper triangular Toeplitz matrices, and $\mathbf{L}'_j \underline{y}$ and $\mathbf{U}_j \underline{x}$ can be regarded as convolutions. The Levinson-Durbin algorithm for inverting a Toeplitz matrix has been generalized to accommodate arbitrary nonsingular matrices via the introduction of a Toeplitz distance concept (e.g. Friedlander, Morf, Kailath, and Ljung [21]; Kailath, Kung, and Morf [50]).

The Toeplitz distance d of a matrix reflects the complexity of the algebraic structure of the matrix considered in relation to the Toeplitz structure. It has been found that the computational efficiency for inverting a given matrix decreases linearly with increase in its Toeplitz distance d . More precisely, with the help of the generalized Levinson algorithm the inversion can be performed at the cost of $(d+2)m^2$ operations, where m is the matrix order.

The results concerning the evaluation of determinant of a symmetric Toeplitz matrix are useful not only for the simplified calculation of the lambda statistics required for the asymptotic likelihood ratio test of a Toeplitz covariance structure, but also in the field of structural engineering for obtaining the m -dimensional volume V of the parallelograms with X_1, X_2, \dots, X_m as edges, since the determinant of the Gramian matrix $X'X$ equals V^2 .

Using the determinant of Σ_m and that of Σ_{m-1} , it can be further shown, following Whittle [70, p. 74], that the statistics

$$\frac{\det(\hat{\Sigma}_m)}{\det(\hat{\Sigma}_{m-1})} = \hat{a}_0 - \hat{\underline{a}}' \hat{\Sigma}_{m-1}^{-1} \hat{\underline{a}}, \quad (8.3)$$

where $\hat{\underline{a}}' = (a_1, a_2, \dots, a_{m-1})$ are the m .l. estimates of the off-diagonal elements of Σ_m and \hat{a}_0 is the m .l. estimate of the common element of the principal diagonal of Σ_m and Σ_{m-1} , can be used for testing the fit of an autoregressive scheme of order $m-1$ against some other order, say m .

It is to be noted further that matrices like Σ_m and their block extensions are members of an m -dimensional quadratic subspace \mathbb{B} of real symmetric matrices as defined by Seely [64]. The subspace \mathbb{B} is quadratic if and only if $A^2 \in \mathbb{B}$ whenever $A \in \mathbb{B}$, i.e., \mathbb{B} is closed under the multiplication $A \circ C = \frac{1}{2}(AC + CA)$. Jensen [44, 45] observes that the latter property makes \mathbb{B} an m -dimensional special Jordan algebra. By Seely's Lemma 2(a), it can be proved that the inverses of these matrices are also members of a quadratic subspace. Thus all of resulting matrices form part of the special Jordan algebra (Jacobson [43]). Hence, the useful properties of Jordan algebra can be fruitfully applied in solving various mathematical and statistical problems connected with the nonsingular symmetric Toeplitz matrix, such as variance components analysis using MINQUE. Work in this line is currently in progress.

REFERENCES

1. A. C. Aitken, *Determinants and Matrices*, Oliver and Boyd, Edinburgh, 1967.
2. M. A. Aitkin, W. C. Nelson, and K. H. Reinfurt, Tests for Correlation Matrices, *Biometrika* 55:327-334 (1968).

- 3 H. Akaike, Block Toeplitz matrix inversion, *SIAM J. Appl. Math.* 24:234–241 (1973).
- 4 T. W. Anderson, Maximum likelihood estimation of parameters of autoregressive processes with moving average residuals and other covariance matrices with linear structure, *Ann. Statist.* 3:1283–1304 (1975).
- 5 E. H. Bereis, Numerical solution of linear equation with Toeplitz and vector Toeplitz matrices, *Numer. Math.* 13:404–424 (1969).
- 6 A. Basilevsky, *Applied Matrix Algebra in the Statistical Sciences*, North Holland, New York, 1983.
- 7 R. Bellman, *Introduction of Matrix Analysis* (2nd ed.), Tata McGraw-Hill, New Delhi, 1974.
- 8 A. Brauer, Limits for the characteristic roots of a matrix VII, *Duke Math. J.* 25:583–590 (1958).
- 9 A. Bottcher and B. Silbermann, *Invertibility and Asymptotics of Toeplitz Matrices*, Mathematical Research, Band 17, Akademie-Verlag, Berlin, 1983.
- 10 K. P. Bube and R. Burridge, The one-dimensional inverse problem of reflection seismology, *SIAM Rev.* 25:497–559 (1983).
- 11 R. E. Caflisch, An inverse problem for Toeplitz matrices and the synthesis of discrete transmission lines, *Linear Algebra Appl.* 38:207–225 (1981).
- 12 A. Calderon, E. Spitzer, and H. Widom, Inversion of Toeplitz matrices, *Illinois J. Math.* 3:490–498 (1959).
- 13 H. E. Daniels, The approximate distribution of serial correlation coefficients, *Biometrika* 43:169–175 (1956).
- 14 K. M. Day, Toeplitz matrices generated by Laurent series expansion of an arbitrary rational function, *Trans. Amer. Math. Soc.* 206:224–245 (1975).
- 15 P. Dewilde, A. C. Vieri, and T. Kailath, On a generalized Szego-Levinson algorithm for optimal linear predictors based on a network synthesis approach, *IEEE Trans. Circuits and Systems* CAS-25; 663–675 (1978).
- 16 B. W. Dickinson, Solution of linear equations with rational Toeplitz matrices, *Math. Comp.* 34:227–233 (1980).
- 17 —, An inverse problem for Toeplitz matrices, *Linear Algebra Appl.* 59:79–83 (1984).
- 18 J. Durbin, Efficient estimation of parameters in moving-average models, *Biometrika* 46:306–316 (1959).
- 19 J. Durbin, The fitting of time series models, *Rev. Internat. Inst. Statist.* 28:233–244 (1966).
- 20 M. Fiedler, Quasidirect decomposition of Hankel and Toeplitz matrices, *Linear Algebra Appl.* 61:155–174 (1984).
- 21 B. Friedlander, M. Morf, T. Kailath, and L. Ljung, New inversion formulas for matrices classified in terms of their distance from Toeplitz matrices, *Linear Algebra Appl.* 27:31–60 (1979).
- 22 G. Frobenius, Über das Tragheitsgesetz der quadratischen Formem, *Sitzungsber. Königl. Preuss. Akad. Wiss.*, 1894, pp. 241–256 (cited in [42]).
- 23 —, Ableitung eines Satzes von Carathéodory aus einer Formel von Kronecker, *Sitzungsber. Königl. Preuss. Akad. Wiss.*, 1912, pp. 16–31 (cited in [42]).

- 24 F. R. Gantmacher, *Applications of the Theory of Matrices*, Interscience, New York, 1959.
- 25 I. C. Gohberg and I. A. Feldman, *Convolution Equations and Projection Methods for their Solutions*, Transl. Math. Monographs, Vol. 41, Amer. Math. Soc., Providence, 1974.
- 26 I. C. Gohberg and G. Heining, Inversion of finite Toeplitz matrices with entries being elements from a noncommutative algebra (in Russian), *Rev. Roumaine Math. Pures Appl.* 19:623–665 (1974).
- 27 I. C. Gohberg and A. A. Semencul, On the inversion of finite Toeplitz matrices and their continuous analogs (in Russian), *Mat. Issled.* 7(2):201–223 (1972).
- 28 I. Gohberg, T. Kailath, and I. Kolitracht, Efficient solution of linear systems of equations with recursive structure, *Linear Algebra Appl.* 80:81–113 (1986).
- 29 M. B. Gorodeckii, Toeplitz determinants generated by rational functions (in Russian), Manuscript deposited in VINITI, No. 5451-81 Dep., Dec. 1981.
- 30 M. J. C. Gover and S. Barnett, Generating polynomials for matrices with Toeplitz or conjugate-Toeplitz matrices, *Linear Algebra Appl.* 61:253–275 (1984).
- 31 F. A. Graybill, *Introduction to Matrices with Application in Statistics*, Wadsworth, Belmont, Calif., 1969.
- 32 U. Grenander and G. Szego, *Toeplitz Forms and Their Applications*, Univ. of California Press, Berkeley, 1958.
- 33 U. Grenander and M. Rosenblatt, *Statistical Analysis of Stationary Time Series*, Wiley, New York, 1957.
- 34 T. N. E. Greville, Toeplitz matrices with Toeplitz inverse revisited, *Linear Algebra Appl.* 5:87–92 (1983).
- 35 T. N. E. Greville and W. F. Trench, Band matrices with Toeplitz inverses, *Linear Algebra Appl.* 27:199–209 (1979).
- 36 F. A. Grunbaum, Toeplitz matrices commuting into tridiagonal matrices, *Linear Algebra Appl.* 40:25–36 (1981).
- 37 F. A. Grunbaum, Eigenvectors of a Toeplitz matrix: Discrete version of prolate spheroidal wave functions, *SIAM J. Algebraic Discrete Methods* 2:136–141 (1981).
- 38 E. V. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, *Linear Algebra Appl.* 1:73–81 (1968).
- 39 G. Heining and K. Rost, *Algebraic Methods for Matrices and Operators*, Mathematical Research, Band 19, Akademie-Verlag, Berlin, 1948; Birkhäuser, Boston, to be published.
- 40 A. J. Hoffman, On the nonsingularity of real matrices, *Math. Comp.* 19:56–61 (1965).
- 41 N. M. Huang and R. E. Cline, Inversion of persymmetric matrices having Toeplitz inverses, *J. Assoc. Comput. Mach.* 19:437–444 (1972).
- 42 I. S. Iohvidov, *Hankel and Toeplitz Matrices and Forms (Algebraic Theory)* (translated by G. Philips and A. Thijse), Birkhäuser, Boston, 1982.
- 43 N. Jacobson, *Structure and Representation of Jordan Algebra*, Amer. Math. Soc., Providence, 1968.
- 44 S. T. Jensen, Covariance hypotheses which are linear in both the covariance and

- the inverse covariance, Preprint No. 1, Inst. of Mathematical Statistics, Univ. of Copenhagen, Jan. 1975.
- 45 S. T. Jensen, An algebraic theory for normal statistical models with unknown covariance, in *Recent Development in Statistics* (J. R. Barra et al., Eds.), North-Holland, Amsterdam, 1977, pp. 285–288.
- 46 K. G. Jöreskog, A general method for analysis of covariance structure, *Biometrika* 57:239–251 (1970).
- 47 J. H. Justice, An algorithm for inverting positive definite Toeplitz matrices, *SIAM J. Appl. Math.* 23:289–291 (1972).
- 48 T. Kailath, A. Buckstein, and D. Morgan, Fast matrix factorization via discrete transmission lines, *Linear Algebra Appl.* 75:1–25 (1986).
- 49 T. Kailath and I. Kolitracht, Matrices with blocks Toeplitz inverses, *Linear Algebra Appl.* 75:145–153 (1986).
- 50 T. Kailath, S. Y. Kung, and M. Morf, Displacement ranks of matrices and linear equations, *J. Math. Anal. Appl.* 68:395–407 (1979).
- 51 T. Kailath, A. Vieira, and M. Morf, Inverses of Toeplitz operators, innovations and orthogonal polynomials, *SIAM Rev.* 20:106–119 (1978).
- 52 N. Levinson, The Wiener RMS (root mean square) error criterion in filter design and prediction, *J. Math. Phys.* 25:261–278 (1947).
- 53 J. Lillestol, Some Statistical Problems Related to Multiple Time Series Models with Structured Spectral Density Matrices, Technical Report, Norwegian School of Economics and Business Administration, Bergen, Norway, 1976.
- 54 R. P. Mentz, On the inverse of some covariance matrices of Toeplitz type, *SIAM J. App. Math.* 31:426–437 (1976).
- 55 B. N. Mukherjee, Likelihood ratio tests of statistical hypotheses associated with patterned covariance matrices in psychology, *British J. Math. Statist. Psych.* 23:87–108 (1970).
- 56 B. N. Mukherjee and S. S. Maiti, Estimation and Testing of Toeplitz Covariance Structures, Technical Report No. ASC/86/9, Applied Statistics, Survey and Computing Division, Indian Statistical Inst., Calcutta, 1986.
- 57 A. Nehorai and M. Morf, A unified derivation for fast estimation algorithms by the conjugate direction method, *Linear Algebra Appl.* 72:119–143 (1985).
- 58 D. V. Ouellet, Schur complements and statistics, *Linear Algebra Appl.* 36:187–195 (1981).
- 59 M. Pagano, An algorithm for fitting autoregressive schemes, *Appl. Statist.* 21:274–281 (1972).
- 60 E. Parzen, An approach to time series analysis, *Ann. Math. Statist.* 32:951–989 (1961).
- 61 ———, Efficient estimation of stationary time series mixed schemes, *Bull. Internat. Statist. Inst.* 44:315–319 (1971).
- 62 N. J. Pullman, *Matrix Theory and Its Applications*, Marcel Dekker, New York, 1976.
- 63 J. Rissanen, Algorithms for triangular decompositions of block Hankel and Toeplitz matrices with applications to factoring positive matrix polynomials, *Math. Comp.* 27:147–154 (1973).

- 64 J. Seely, Quadratic subspaces and completeness, *Ann. Math. Statist.* 42:710–721 (1971).
- 65 M. M. Siddiqui, On the inversion of the sample covariance matrix in a stationary autoregressive process, *Ann. Math. Statist.* 29:585–588 (1958).
- 66 W. F. Trench, An algorithm for the inversion of finite Toeplitz matrices, *SIAM J. Appl. Math.* 12:515–522 (1964).
- 67 ———, Weighing coefficients for the prediction of stationary time series from the finite past, *SIAM J. Appl. Math.* 15:1502–1510 (1967).
- 68 ———, On the eigenvalue problem for Toeplitz matrices generated by rational functions, *Linear and Multilinear Algebra* 17:337–353 (1985).
- 69 ———, Solution of systems with Toeplitz matrices generated by rational functions, *Linear Algebra Appl.* 74:191–211 (1986).
- 70 P. Whittle, *Hypothesis Testing in Time Series Analysis*, Almqvist and Wiksells, Uppsala, Sweden, 1951.
- 71 H. Widom, Inversion of Toeplitz matrices II, *Illinois J. Math.* 4:88–99 (1960).
- 72 B. J. Winer, *Statistical Principles in Experimental Design*, McGraw-Hill, New York, 1962.
- 73 J. Wise, The autocorrelation function and the spectral density function, *Biometrika* 42:151–159 (1955).
- 74 P. Zellini, On the optimal computation of a set of symmetric and persymmetric bilinear forms, *Linear Algebra Appl.* 23:101–119 (1979).
- 75 S. Zohar, The algorithm of W. F. Trench, *J. Assoc. Comput. Mach.* 16:592–601 (1969).
- 76 S. Zohar, The solution of a Toeplitz set of linear equations. *J. Assoc. Comput. Mach.* 21:272–276 (1974).

Received 11 January 1987; final manuscript accepted 11 June 1987